

ON SETS OF MARKED ONCE-HOLED TORI ALLOWING HOLOMORPHIC MAPPINGS INTO RIEMANN SURFACES WITH MARKED HANDLE

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ABSTRACT. In our previous work [10], for a given Riemann surface Y_0 with marked handle, we investigated geometric properties of the set of marked once-holed tori X allowing holomorphic mappings of X into Y_0 . It turned out that it is a closed domain with Lipschitz boundary. In the present paper we show that the boundary is never smooth. Also, we evaluate the critical extremal length for the existence of holomorphic mappings in terms of hyperbolic lengths.

1. INTRODUCTION

Let R_1 and R_2 be Riemann surfaces. It is a natural question whether there are holomorphic or conformal mappings of R_1 into R_2 with some geometric or analytic properties. In the present article we consider the problem in the case where R_1 is a once-holed torus, and look for handle-preserving mappings.

Since Riemann surfaces of genus zero are conformally equivalent to plane regions, Riemann surfaces of positive genus should play the leading character in Riemann surfaces theory. Once-holed tori are topologically the simplest among the nonplanar Riemann surfaces. They are building blocks of Riemann surfaces of positive genus; every Riemann surface of positive genus g is obtained from g once-holed tori by suitable identification. Open disks are one of the simplest plane domains, and studies of functions on open disks are of fundamental importance for local theory. Thus studies of holomorphic mappings on once-holed tori would be significant for “local theory” of holomorphic mappings between Riemann surfaces. While open disk are conformally equivalent to one another, once-holed tori are not. Hence we need to know which once-holed tori are included in a Riemann surface under consideration. This amounts to ask the existence of conformal mappings of once-holed tori into Riemann surfaces.

For the existence of conformal mappings of once-holed tori several results are known. In [11] and [12] Shiba investigated the set of tori into which a given open Riemann surface of genus one is conformally embedded. His results give solutions to our problem in the case where R_2 is a torus. Also, we gave a characterization for the existence of conformal mappings of a once-holed torus into another explicitly in terms of finitely many extremal lengths (see [7]). In [8] we examined the set of once-holed tori that can be conformally embedded into a given Riemann surface

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of positive genus. For topologically finite surfaces Kahn-Pilgrim-Thurston [4] has recently given a characterization for the existence of conformal embeddings in terms of extremal lengths.

On the other hand, few results are known for the existence of holomorphic mappings of once-holed tori. If R_2 is a torus, then the Behnke-Stein theorem yields that any once-holed torus allows handle-preserving holomorphic mappings into R_2 . If R_2 is not a torus, then it carries a hyperbolic metric. Since holomorphic mappings decrease hyperbolic lengths, we obtain necessary conditions for the existence of holomorphic mappings. However, they are not sufficient by a recent result of Bourque [1].

The space \mathfrak{T} of marked once-holed tori is a three-dimensional real-analytic manifold with boundary. In our previous work [10], for a given Riemann surface Y_0 with marked handle, we investigated the set $\mathfrak{T}_a[Y_0]$ of marked once-holed tori X such that there is a holomorphic mapping of X into Y_0 . We introduced a new condition called a handle condition to obtain the following two results.

Proposition 1 ([10, Theorem 1]). *$\mathfrak{T}_a[Y_0]$ is a closed domain with Lipschitz boundary, and is a retract of the whole space \mathfrak{T} .*

The second result is expressed in terms of a specific coordinate system on \mathfrak{T} . Every once-holed torus is realized as a slit torus. For $(\tau, s) \in \mathbb{H} \times [0, 1]$ let $X_\tau^{(s)}$ denote the marked once-holed torus obtained from the marked torus X_τ of modulus τ by deleting a horizontal segment of length s , where \mathbb{H} is the upper half-plane.

Proposition 2 ([10, Theorem 2]). *There exists a nonnegative number $\lambda_a[Y_0]$ such that*

- (i) *if $\operatorname{Im} \tau \geq 1/\lambda_a[Y_0]$, then there are no holomorphic mappings of $X_\tau^{(s)}$ into Y_0 for any s , while*
- (ii) *if $\operatorname{Im} \tau < 1/\lambda_a[Y_0]$, then there are holomorphic mappings of $X_\tau^{(s)}$ into Y_0 for some s .*

Propositions 1 and 2 raise the following natural questions:

- (1) Is the boundary of $\mathfrak{T}_a[Y_0]$ smooth?
- (2) What is the value of $\lambda_a[Y_0]$?

In the present paper we answer these questions. We first show that the boundary of $\mathfrak{T}_a[Y_0]$ is not smooth in most cases:

Theorem 1. *If Y_0 is not a marked torus or a marked once-punctured torus, then the boundary of $\mathfrak{T}_a[Y_0]$ is not smooth.*

We prove Theorem 1 in §3 after summarizing results of [10] in §2. In §4 we compare $\mathfrak{T}_a[Y_0]$ with the set $\mathfrak{T}_\sigma[Y_0]$ of marked once-holed tori having longer geodesics than corresponding geodesics on Y_0 . In the final section we give an answer to second question (2) (see Theorem 3).

2. PRELIMINARIES

Let R be a Riemann surface of positive genus; it may be compact or of infinite genus. It has one or more handles. A handle of R is specified by a couple of loops on R . With this in mind we make the following definitions. A *mark of handle* of R is, by definition, an ordered pair $\chi = \{a, b\}$ of simple loops a and b on R whose geometric intersection number $a \times b$ is equal to one. A *Riemann surface with marked*

handle means a pair (R, χ) , where R is a Riemann surface of positive genus and χ is a mark of handle of R .

Let $Y_1 := (R_1, \chi_1)$ and $Y_2 := (R_2, \chi_2)$ be Riemann surfaces with marked handle, where $\chi_j = \{a_j, b_j\}$ for $j = 1, 2$. If a continuous mapping $f : R_1 \rightarrow R_2$ maps a_1 and b_1 onto loops freely homotopic to a_2 and b_2 on R_2 , respectively, then we say that f is a continuous mapping of Y_1 into Y_2 and use the notation $f : Y_1 \rightarrow Y_2$. If $f : R_1 \rightarrow R_2$ possesses some additional properties, then $f : Y_1 \rightarrow Y_2$ is said to possess the same properties. For example, if $f : R_1 \rightarrow R_2$ is conformal, then $f : Y_1 \rightarrow Y_2$ is called conformal. Here, by a conformal mapping we mean a holomorphic injection; we do not require conformal mappings to be surjective. We consider continuous mappings of Y_1 into Y_2 preserve the handles specified by χ_1 and χ_2 .

A *once-holed torus* is, by definition, a noncompact Riemann surface of genus one with exactly one boundary component in the sense of Kerékjártó-Stoilow. For example, the Riemann surface obtained from a torus, that is, a compact Riemann surface of genus one, by removing one point is a once-holed torus, which will be referred to as a *once-punctured torus*. Note that once-holed tori are not bordered surfaces. A once-holed torus with marked handle is usually called a *marked once-holed torus*. The meaning of a *marked once-punctured torus* is obvious.

Let \mathfrak{T} denote the set of marked once-holed tori, where two marked once-holed tori are identified if there is a conformal mapping of one *onto* the other. As a set, it is the disjoint union of the Teichmüller space \mathfrak{T}' of a once-punctured torus and the reduced Teichmüller space \mathfrak{T}'' of a once-holed torus that is not a once-punctured torus.

There is a canonical injection Λ of \mathfrak{T} into \mathbb{R}_+^3 ; if $X = (T, \chi)$ with $\chi = \{a, b\}$, then $\Lambda(X)$ is the triplet of the extremal lengths of the free homotopy classes of a , b and ab^{-1} . We know that

$$\mathfrak{L} := \Lambda(\mathfrak{T}) = \{\mathbf{x} \in \mathbb{R}_+^3 \mid Q(\mathbf{x}) + 4 \leq 0\},$$

where $Q(\mathbf{x}) = Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2(x_1x_2 + x_2x_3 + x_3x_1)$. Note that Λ maps \mathfrak{T}' and \mathfrak{T}'' onto the boundary $\partial\mathfrak{L}$ and the interior $\mathfrak{L}^\circ := \mathfrak{L} \setminus \partial\mathfrak{L}$, respectively. Moreover, the restrictions $\Lambda|_{\mathfrak{T}'} : \mathfrak{T}' \rightarrow \partial\mathfrak{L}$ and $\Lambda|_{\mathfrak{T}''} : \mathfrak{T}'' \rightarrow \mathfrak{L}^\circ$ are real-analytic diffeomorphisms. We regard \mathfrak{T} as a three-dimensional real-analytic manifold with boundary so that $\Lambda : \mathfrak{T} \rightarrow \mathfrak{L}$ is a real-analytic diffeomorphism. In the rest of the article we use the notations $\partial\mathfrak{T}$ and \mathfrak{T}° instead of \mathfrak{T}' and \mathfrak{T}'' , respectively. For details, see [7, §7].

Now, fix a Riemann surface $Y_0 = (R_0, \chi_0)$ with marked handle. For a given marked once-holed torus X there may or may not exist holomorphic mappings of X into Y_0 . We are interested in the set of marked once-holed tori which allow holomorphic mappings into Y_0 . We denote by $\mathfrak{T}_a[Y_0]$ (resp. $\mathfrak{T}_c[Y_0]$) the set of $X \in \mathfrak{T}$ such that there exists a holomorphic (resp. conformal) mapping of X into Y_0 . In our previous work [10] we introduced handle conditions to investigated geometric properties of $\mathfrak{T}_a[Y_0]$ and $\mathfrak{T}_c[Y_0]$. We recall the definition of a handle condition.

For $X, X' \in \mathfrak{T}$ we say that X is smaller than X' and write $X \preceq X'$ if there is a conformal mapping of X into X' . The relation \preceq is then an order relation on \mathfrak{T} .

A mathematical statement $\mathcal{P}(X)$, where the free variable X ranges over \mathfrak{T} , is called a *handle condition* if $\mathcal{P}(X_1)$ implies $\mathcal{P}(X_2)$ for all $X_1, X_2 \in \mathfrak{T}$ with $X_2 \preceq X_1$. Important examples are the statements “there is a holomorphic mapping of X into Y_0 ” and “there is a conformal mapping of X into Y_0 ,” which will be denoted by

$\mathcal{P}_a(X)$ and $\mathcal{P}_c(X)$, respectively. For $\nu \in \mathbb{N}$ the statement

$\mathcal{P}_\nu(X)$ = “There is a holomorphic mapping $f : X \rightarrow Y_0$ with $d(f) < \nu + 1$ ”

is another handle condition, where $d(f)$ is the supremum of the cardinal numbers of $f^{-1}(p)$, $p \in R_0$. Note that $\mathcal{P}_1(X) = \mathcal{P}_c(X)$.

Set $\mathfrak{T}[\mathcal{P}] = \{X \in \mathfrak{T} \mid \mathcal{P}(X)\}$. Then we have the following proposition.

Proposition 3 ([10, Theorem 3]). *If $\mathfrak{T}[\mathcal{P}] \neq \emptyset$, then its interior $\mathfrak{T}^\circ[\mathcal{P}]$ is a domain with Lipschitz boundary.*

Remark. In the case where $\mathfrak{T}[\mathcal{P}] = \mathfrak{T}$, we consider $\mathfrak{T}[\mathcal{P}]$ to have a Lipschitz boundary though the boundary $\partial\mathfrak{T}[\mathcal{P}]$ is in fact an empty set.

Actually, we can show more. The eigenvalues of the coefficient matrix of the quadratic form $Q(\mathbf{x})$ are -1 and 2 . The corresponding eigenspaces V_{-1} and V_2 are, respectively, the line $x_1 = x_2 = x_3$ and the plane $x_1 + x_2 + x_3 = 0$. Let $\mathbf{e} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}) \in V_{-1}$.

Proposition 4 ([10, Proposition 4]). *There is a Lipschitz continuous function $e[\mathcal{P}](\cdot)$ on V_2 such that*

$$\Lambda(\mathfrak{T}^\circ[\mathcal{P}]) = \{\zeta + t\mathbf{e} \mid \zeta \in V_2[\mathcal{P}], t > e[\mathcal{P}](\zeta)\},$$

provided that $\mathfrak{T}[\mathcal{P}] \neq \emptyset$.

Since $\mathfrak{T}_a[Y_0] = \mathfrak{T}[\mathcal{P}_a]$, Proposition 1 follows from Proposition 3 together with the fact that $\mathfrak{T}_a[Y_0]$ is closed. The set $\mathfrak{T}_c[Y_0] := \mathfrak{T}[\mathcal{P}_c]$ possesses the same properties. In fact, setting $\mathfrak{T}_\nu[Y_0] = \mathfrak{T}[\mathcal{P}_\nu]$, we deduce the following proposition.

Proposition 5 ([10, Corollary 1]). *The sets $\mathfrak{T}_\nu[Y_0]$, $\nu \in \mathbb{N}$, are closed domains with Lipschitz boundary, and are retracts of \mathfrak{T} .*

Every marked once-holed torus is realized as a horizontal slit torus (see Shiba [11]). Specifically, for each point τ in the upper half-plane \mathbb{H} , let G_τ be the additive subgroup of \mathbb{C} generated by 1 and τ . Then $T_\tau := \mathbb{C}/G_\tau$ is a torus. Let $\pi_\tau : \mathbb{C} \rightarrow T_\tau$ be the natural projection, and set $a_\tau = \pi_\tau([0, 1])$ and $b_\tau = \pi_\tau([0, \tau])$, where $[z_1, z_2]$ stands for the oriented line segment joining z_1 with z_2 ; if $z_1 = z_2$, then $[z_1, z_2]$ denotes the singleton $\{z_1\}$. Then $\chi_\tau := \{a_\tau, b_\tau\}$ is a mark of handle of T_τ , and we obtain a marked torus $X_\tau := (T_\tau, \chi_\tau)$.

Now, for $s \in [0, 1)$ set $T_\tau^{(s)} = T_\tau \setminus \pi_\tau([0, s])$; it is a once-holed torus. We choose a mark $\chi_\tau^{(s)} = \{a_\tau^{(s)}, b_\tau^{(s)}\}$ of handle of $T_\tau^{(s)}$ so that the inclusion mapping of $T_\tau^{(s)}$ into T_τ is a conformal mapping of $X_\tau^{(s)} := (T_\tau^{(s)}, \chi_\tau^{(s)})$ into X_τ . Then the correspondence $(\tau, s) \mapsto X_\tau^{(s)}$ is a homeomorphism of $\mathbb{H} \times [0, 1)$ onto \mathfrak{T} , which is a real-analytic diffeomorphism on $\mathbb{H} \times (0, 1)$ (see [10, §4]). In other words, its inverse $\Sigma : X_\tau^{(s)} \mapsto (\tau, s)$ serves as a global topological coordinate system on \mathfrak{T} .

Let $\Pi : \mathbb{H} \times [0, 1) \rightarrow \mathbb{H}$ be the natural projection. Then for any handle condition $\mathcal{P}(X)$ the image $\mathbb{H}[\mathcal{P}] := \Pi \circ \Sigma(\mathfrak{T}[\mathcal{P}])$ is a horizontal strip. To be more precise for $t \in \mathbb{R}_+ := \mathbb{R}_+ \cup \{+\infty\} = [0, +\infty]$ set $\mathcal{H}(t) = \{t \in \mathbb{C} \mid 0 < \operatorname{Im} z < t\}$ and let $\bar{\mathcal{H}}(t)$ denote its closure in \mathbb{H} . Note that $\mathcal{H}(0) = \bar{\mathcal{H}}(0) = \emptyset$ and $\mathcal{H}(+\infty) = \bar{\mathcal{H}}(+\infty) = \mathbb{H}$.

Proposition 6 ([10, Theorem 4]). *For every handle condition $\mathcal{P}(X)$ there exists a constant $\lambda[\mathcal{P}] \in \mathbb{R}_+$ such that*

$$\mathcal{H}\left(\frac{1}{\lambda[\mathcal{P}]}\right) \subset \mathbb{H}[\mathcal{P}] \subset \bar{\mathcal{H}}\left(\frac{1}{\lambda[\mathcal{P}]}\right),$$

where $1/0 = +\infty$ and $1/(+\infty) = 0$.

We set $\lambda_a[Y_0] = \lambda[\mathcal{P}_a]$ and $\lambda_c[Y_0] = \lambda[\mathcal{P}_c]$. They are referred to as the *critical extremal lengths* for the existence of holomorphic and conformal mappings of marked once-holed tori into Y_0 , respectively. Most part of Proposition 2 follows from Proposition 6. Note, however, that Proposition 2 asserts that the identity $\mathbb{H}[\mathcal{P}_a] = \mathcal{H}(1/\lambda[\mathcal{P}_a])$ actually holds. On the other hand, in general, $\mathcal{H}(1/\lambda[\mathcal{P}_c])$ is a proper subset of $\mathbb{H}[\mathcal{P}_c]$ (see [10, Example 13]).

3. NON-SMOOTHNESS OF BOUNDARIES

Propositions 1 and 5 show that $\mathfrak{T}_a[Y_0]$ and $\mathfrak{T}_c[Y_0]$ have Lipschitz boundaries. It is then natural to ask whether the boundaries are smooth or not. As for $\mathfrak{T}_c[Y_0] = \mathfrak{T}_1[Y_0]$ we know that the answer is negative in general. In fact, if Y_0 is a marked once-holed torus, then $\Lambda(\mathfrak{T}_c[Y_0])$ is a cone with vertex at $\Lambda(Y_0)$ and hence the boundary of $\mathfrak{T}_c[Y_0]$ is not smooth at Y_0 (see [10, Example 10]). Our first result, Theorem 1, claims that the boundary of $\mathfrak{T}_a[Y_0]$ is not, either, for most cases.

For the proof of Theorem 1 we define the handle covering surface of a Riemann surface Y_0 with marked handle. There is a Riemann surface $\tilde{Y}_0 = (\tilde{R}_0, \tilde{\chi}_0)$ with marked handle together with a holomorphic mapping $\pi_0 : \tilde{Y}_0 \rightarrow Y_0$ such that

- (i) the fundamental group of \tilde{R}_0 is generated by the loops in $\tilde{\chi}_0$, and
- (ii) $\pi_0 : \tilde{R}_0 \rightarrow R_0$ is a covering map.

We call \tilde{Y}_0 the *handle covering surface* of Y_0 . The following lemma is easily verified.

Lemma 1. *Let Y_0 be a Riemann surface of marked handle and \tilde{Y}_0 its handle covering surface.*

- (i) *If Y_0 is not a marked torus, then \tilde{Y}_0 is a marked once-holed torus. If Y_0 is a marked torus or a marked once-holed torus, then $\tilde{Y}_0 = Y_0$.*
- (ii) *If \tilde{Y}_0 is a marked once-punctured torus, then so is Y_0 .*
- (iii) $\tilde{Y}_0 \in \mathfrak{T}_a[Y_0]$.
- (iv) $\mathfrak{T}_a[\tilde{Y}_0] = \mathfrak{T}_a[Y_0]$.

Proof of Theorem 1. By Lemma 1 (iv) we may assume from the outset that Y_0 is an element of \mathfrak{T}° . We employ Fenchel-Nielsen coordinates (see Buser [2]). To be specific, let $X = (T, \chi) \in \mathfrak{T}$, where $\chi = \{a, b\}$. The once-holed torus T carries a hyperbolic metric, whose curvature is normalized to be -1 . Denote by $l(X)$ the length of the hyperbolic geodesic α freely homotopic to a . Let $\theta(X)$ stand for the twist parameter along α . Also, let $l'(X)$ be the infimum of hyperbolic lengths of loops freely homotopic to $aba^{-1}b^{-1}$. Clearly, $l'(X)$ vanishes if and only if X is a marked once-punctured torus. Setting $\Phi(X) = (l(X), l'(X), \theta(X))$, we obtain a homeomorphism of \mathfrak{T} onto $(0, +\infty) \times [0, +\infty) \times \mathbb{R}$, which is a real-analytic diffeomorphism of \mathfrak{T}° onto $(0, +\infty) \times (0, +\infty) \times \mathbb{R}$.

If $X \in \mathfrak{T}_a[Y_0]$, then $l(X) \geq l(Y_0)$ and $l'(X) \geq l'(Y_0)$ since holomorphic mappings decrease hyperbolic metrics. As $Y_0 \in \mathfrak{T}_a[Y_0]$, these inequalities imply that Y_0 lies on the boundary $\partial\mathfrak{T}_a[Y_0]$ and that $\partial\mathfrak{T}_a[Y_0]$ is not smooth at Y_0 , provided that $Y_0 \notin \partial\mathfrak{T}$. Theorem 1 has been thus established. \square

Remark. If Y_0 is a marked torus, then $\mathfrak{T}_a[Y_0]$ coincides with the whole space \mathfrak{T} (see [10, Example 9]). Thus its boundary is an empty set. For marked once-punctured tori Y_0 the boundary of $\Phi(\mathfrak{T}_a[Y_0])$ is not smooth. However, we do not know whether the boundary of $\Lambda(\mathfrak{T}_a[Y_0])$ is smooth or not.

It is obvious that

$$\mathfrak{T}_c[Y_0] = \mathfrak{T}_1[Y_0] \subset \cdots \subset \mathfrak{T}_\nu[Y_0] \subset \mathfrak{T}_{\nu+1}[Y_0] \subset \cdots \subset \mathfrak{T}_a[Y_0].$$

If $Y_0 \in \mathfrak{T}^\circ$, then both of $\partial\mathfrak{T}_c[Y_0]$ and $\partial\mathfrak{T}_a[Y_0]$ contains Y_0 and are not smooth at Y_0 . We thus have the following corollary to Theorem 1.

Corollary 1. *If Y_0 is a marked once-holed torus which is not a marked once-punctured torus, then for any positive integer ν the boundary of $\mathfrak{T}_\nu[Y_0]$ is not smooth.*

4. HYPERBOLIC LENGTH SPECTRA

Let W be a free group generated by two elements. We regard it as the set of reduced words $w(u, v)$ of two letters u and v . The unit is the void word. We denote by W^* the subset of non-unit elements.

In general, let $Y = (R, \chi)$, where $\chi = \{a, b\}$, be a Riemann surface with marked handle. For $w = w(u, v) \in W^*$ the notation $w(a, b)$ denotes a loop on R . We set $w(Y) = w(a, b)$. In particular, $u(Y) = a$ and $v(Y) = b$. Let $l(Y, w)$ be the infimum of the hyperbolic lengths of loops in the free homotopy class $\Gamma(Y, w)$ of $w(Y)$ on R , provided that R is not a torus. In the case where R is a torus, we set $l(Y, w) = 0$ for convenience.

Now, fix a Riemann surface Y_0 with marked handle, and let $\mathfrak{T}_\sigma[Y_0]$ be the set of $X \in \mathfrak{T}$ for which $l(X, w) \geq l(Y_0, w)$ for all $w \in W^*$. Since holomorphic mappings decrease hyperbolic lengths, $\mathfrak{T}_a[Y_0]$ is included in $\mathfrak{T}_\sigma[Y_0]$. It follows from Bourque [1] that $\mathfrak{T}_a[Y_0]$ is in general a proper subset of $\mathfrak{T}_\sigma[Y_0]$. The following theorem claims more:

Theorem 2. *Let Y_0 be a Riemann surface with marked handle which is not a marked torus. Then*

- (i) $\mathfrak{T}_\sigma[Y_0]$ is a closed domain with Lipschitz boundary,
- (ii) its boundary $\partial\mathfrak{T}_\sigma[Y_0]$ meets $\mathfrak{T}_a[Y_0]$ exactly at one point, and
- (iii) $\mathfrak{T}_\sigma[Y_0] \setminus \mathfrak{T}_a[Y_0]$ is homeomorphic to $\mathbb{C}^* \times [0, 1)$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Remark. If Y_0 is a marked torus, then $\mathfrak{T}_\sigma[Y_0] = \mathfrak{T}_a[Y_0] = \mathfrak{T}$ (see [10, Example 12]).

For the proof of Theorem 2 we introduce some notations, and prepare several lemmas. We first remark the following lemma.

Lemma 2. *If \tilde{Y}_0 is the handle covering surface of Y_0 , then $\mathfrak{T}_\sigma[\tilde{Y}_0] = \mathfrak{T}_\sigma[Y_0]$.*

Let D be a doubly connected Riemann surface. Denote by $\lambda(D)$ the extremal length of the free homotopy class $\Gamma(D)$ of a simple loop separating the boundary components of D . Unless D is conformally equivalent to the punctured plane $\mathbb{C} \setminus \{0\}$, it carries a hyperbolic metric. Let $l(D)$ stand for the infimum of the hyperbolic lengths of loops in $\Gamma(D)$. Define $l(D) = 0$ if D is conformally equivalent to $\mathbb{C} \setminus \{0\} = 0$. Note that the identity

$$(1) \quad \lambda(D) = \frac{1}{\pi} l(D)$$

holds.

Remark. If a is a simple loop on D separating the boundary components of D , then so is a^{-1} . Though the free homotopy classes Γ^+ and Γ^- of a and a^{-1} , respectively, are disjoint, their extremal lengths are identical with each other. The common value is denoted by $\lambda(D)$. In the sequel $\Gamma(D)$ will represent one of Γ^+ and Γ^- .

Let $Y = (R, \chi)$ be a Riemann surface with marked handle, and take $w \in W^*$. Let D be the annular covering surface of R with respect to the loop $w(Y)$ (see [5, §3]). Thus D is a doubly connected Riemann surface, and there is a holomorphic covering map π of D onto R which maps $\Gamma(D)$ into $\Gamma(Y, w)$. Clearly, we have

$$(2) \quad l(D) = l(Y, w).$$

Lemma 3 (Wolpert [14, Lemma 3.1]). *Let Y_1 and Y_2 be Riemann surfaces with marked handle. If there is a K -quasiconformal mapping of Y_1 onto Y_2 , then*

$$\frac{1}{K}l(Y_1, w) \leq l(Y_2, w) \leq Kl(Y_1, w)$$

for all $w \in W^*$.

In fact, let D_j be the annular covering surface of Y_j with respect to $w(Y_j)$. If there is a K -quasiconformal mapping of Y_1 onto Y_2 , then it is lifted to a K -quasiconformal mapping D_1 onto D_2 which maps $\Gamma(D_1)$ to $\Gamma(D_2)$. Since extremal lengths are quasi-invariant under quasiconformal mappings, the lemma is an immediate consequence of (1) and (2).

Lemma 4. *For each $w \in W^*$ the function $l(\cdot, w)$ is continuous on \mathfrak{T} .*

Proof. It follows from Lemma 3 that $l(\cdot, w)$ is continuous on \mathfrak{T}° . To show that it is also continuous at each point of $\partial\mathfrak{T}$, take an arbitrary marked once-punctured torus $X_{\tau_0}^{(0)}$. For $s \in [0, 1)$ the annular covering surface of $X_{\tau_0}^{(s)}$ with respect to the loop $w(X_{\tau_0}^{(s)})$ is conformally equivalent to the annulus $A(s) := \{z \in \mathbb{C} \mid \exp(-2\pi^2/l(X_{\tau_0}^{(s)}, w)) < |z| < 1\}$. The inclusion mapping ι_s of $X_{\tau_0}^{(s)}$ into $X_{\tau_0}^{(0)}$ induces a conformal mapping of $A(s)$ into $A(0)$. Observe that the function $s \mapsto l(X_{\tau_0}^{(s)}, w)$ is increasing. Since ι_s tends to the identity mapping of $X_{\tau_0}^{(0)}$ onto itself as $s \rightarrow 0$, we see that $l(X_{\tau_0}^{(s)}, w)$ converges to $l(X_{\tau_0}^{(0)}, w)$.

Now, for $(\tau, s) \in \mathbb{H} \times [0, 1)$ the \mathbb{R} -linear mapping F_τ of \mathbb{C} onto itself with $F_\tau(1) = 1$ and $F_\tau(\tau_0) = \tau$ induces a quasiconformal mapping of $X_{\tau_0}^{(s)}$ onto $X_\tau^{(s)}$ whose maximal dilatation is equal to $e^{d_{\mathbb{H}}(\tau_0, \tau)}$, where $d_{\mathbb{H}}(\tau_0, \tau)$ is the distance between τ_0 and τ with respect to the hyperbolic metric on \mathbb{H} . We apply Lemma 3 to obtain

$$\begin{aligned} & |l(X_\tau^{(s)}, w) - l(X_{\tau_0}^{(0)}, w)| \\ & \leq |l(X_\tau^{(s)}, w) - l(X_{\tau_0}^{(s)}, w)| + |l(X_{\tau_0}^{(s)}, w) - l(X_{\tau_0}^{(0)}, w)| \\ & \leq (e^{d_{\mathbb{H}}(\tau_0, \tau)} - e^{-d_{\mathbb{H}}(\tau_0, \tau)})l(X_{\tau_0}^{(s)}, w) + |l(X_{\tau_0}^{(s)}, w) - l(X_{\tau_0}^{(0)}, w)|. \end{aligned}$$

Consequently, $l(X_\tau^{(s)}, w) \rightarrow l(X_{\tau_0}^{(0)}, w)$ as $(\tau, s) \rightarrow (\tau_0, 0)$, which means that the function $l(\cdot, w)$ is continuous at $X_{\tau_0}^{(0)}$, as desired. \square

Corollary 2. $\mathfrak{T}_\sigma[Y_0]$ is a closed subset of \mathfrak{T} .

Proof of Theorem 2. Since conformal mappings decrease hyperbolic metrics, the statement

$$\mathcal{P}_\sigma(X) := "l(X, w) \geq l(Y_0, w) \text{ for all } w \in W^*" "$$

is a handle condition (see [10, Example 8]). As $\mathfrak{T}[P_\sigma] = \mathfrak{T}_\sigma[Y_0]$, Proposition 3 together with Corollary 2 implies assertion (i).

In order to show assertion (ii), by Lemmas 1 (iv) and 2, we have only to consider the case where Y_0 is a marked once-holed torus, say, $X_{\tau_0}^{(s_0)}$. If there is a holomorphic

mapping of a marked once-punctured torus $X_\tau^{(0)}$ into $Y_0 = X_{\tau_0}^{(s_0)}$, then it is extended to a holomorphic mapping between the marked tori X_τ and X_{τ_0} , which must be conformal. Consequently, Y_0 is also a marked once-punctured torus identical with $X_\tau^{(0)}$. We have shown that $\mathfrak{T}_a[Y_0] \cap \partial\mathfrak{T}$ is empty or a singleton and that in the latter case $\mathfrak{T}_a[Y_0] \cap \partial\mathfrak{T}$ consists only of Y_0 .

Take an arbitrary $X \in \mathfrak{T}_a[Y_0] \setminus \{Y_0\}$. We employ arguments in [1] to prove that X lies in the interior of $\mathfrak{T}_\sigma[Y_0]$. Let f be a holomorphic mapping of X into Y_0 . Let $\rho_X = \rho_X(z)|dz|$ and $\rho_{Y_0} = \rho_{Y_0}(\zeta)|d\zeta|$ denote the hyperbolic metrics on X and Y_0 , respectively. By Schwarz's lemma the continuous function $(f^*\rho_{Y_0})/\rho_X$ is strictly less than one pointwise, where $f^*\rho_{Y_0}$ stands for the pull-back of ρ_{Y_0} by f . The convex core C of X is compact and hence there is $c \in (0, 1)$ for which $(f^*\rho_{Y_0})/\rho_X < c$ on C . For $w \in W^*$ let γ_w be the closed geodesic on X freely homotopic to $w(X)$. Since γ_w lies in C , we have

$$l(Y, w) \leq \int_{f_*\gamma_w} \rho_{Y_0} = \int_{\gamma_w} f^*\rho_{Y_0} \leq c \int_{\gamma_w} \rho_X = cl(X, w).$$

There is a neighborhood U of X such that for any $X' \in U$ there is a $(1/c)$ -quasiconformal mapping of X onto X' . Applying Lemma 3, we infer that $X' \in \mathfrak{T}_\sigma[Y_0]$. Thus $U \subset \mathfrak{T}_\sigma[Y_0]$, or, X is an interior point of $\mathfrak{T}_\sigma[Y_0]$, as claimed. We have proved assertion (ii).

Assertion (iii) is now an easy consequence of Proposition 4. This completes the proof. \square

Remark. We see from the proof that the element in $\mathfrak{T}_a[Y_0] \cap \partial\mathfrak{T}_\sigma[Y_0]$ is the handle covering surface \tilde{Y}_0 of Y_0 . There is exactly one holomorphic mapping of \tilde{Y}_0 into Y_0 (see [9]). Therefore, if Y_0 is not a marked once-holed torus, then there are no holomorphic mappings $f : \tilde{Y}_0 \rightarrow Y_0$ with $d(f) < \infty$.

5. CRITICAL EXTREMAL LENGTHS

The purpose of this section is to evaluate the critical extremal lengths for the existence of holomorphic and conformal mappings of marked once-holed tori into a Riemann surface with marked handle. Let $Y = (R, \chi)$, where $\chi = \{a, b\}$, be a Riemann surface with marked handle. Recall that W is the free group generated by u and v . Set $\Gamma(Y) = \Gamma(Y, u)$ and $l(Y) = l(Y, u)$. Thus $\Gamma(Y)$ is the free homotopy class of $a = u(Y)$. Let $\lambda(Y)$ stand for its extremal length. Note that $\lambda(X_\tau^{(s)}) = 1/\text{Im } \tau$.

Now, fix a Riemann surface Y_0 with marked handle. We begin with evaluating the critical extremal length $\lambda_a[Y_0]$ for the existence of holomorphic mappings.

Theorem 3. $\lambda_a[Y_0] = \frac{1}{\pi}l(Y_0)$.

Proof. If Y_0 is a marked torus, then $\mathfrak{T}_a[Y_0] = \mathfrak{T}$ (see the remark following Theorem 2) and hence $\lambda_a[Y_0] = 0$. Since $l(Y_0) = 0$ by definition, we see that the theorem is valid in this case.

Next suppose that Y_0 is not a marked torus. By Lemma 1 (iv) we have only to consider the case where Y_0 is a marked once-holed torus. Let $X_\tau^{(s)}$ be an arbitrary element of $\mathfrak{T}_a[Y_0]$. The image D_τ of the horizontal strip $\{z \in \mathbb{C} \mid 0 < \text{Im } z < \text{Im } \tau\}$

by the projection $\pi_\tau : \mathbb{C} \rightarrow T_\tau = \mathbb{C}/G_\tau$ is a doubly connected domain on T_τ and is included in $T_\tau^{(s)}$. Since $\lambda(X_\tau^{(s)}) = \lambda(D_\tau) = 1/\text{Im } \tau$, we see from (1) that

$$\lambda(X_\tau^{(s)}) = \frac{1}{\pi} l(D_\tau) > \frac{1}{\pi} l(X_\tau^{(s)})$$

(cf. Maskit [6, Proposition 1]). As holomorphic mappings decrease hyperbolic lengths, we obtain $\lambda(X_\tau^{(s)}) > (1/\pi)l(Y_0)$, which implies

$$\lambda_a[Y_0] \geq \frac{1}{\pi} l(Y_0).$$

To show the opposite inequality we employ the annular covering surface D_0 of R_0 with respect to the loop a_0 , where $Y_0 = (R_0, \chi_0)$ and $\chi_0 = \{a_0, b_0\}$. For any $\varepsilon > 0$ choose a doubly connected and relatively compact subdomain D of D_0 with $\Gamma(D) \subset \Gamma(D_0)$ so that

$$l(D) < l(D_0) + \varepsilon = l(Y_0) + \varepsilon.$$

We further assume that the components of ∂D are simple loops on D_0 . Let $\pi_0 : D_0 \rightarrow R_0$ be the covering map. Since the closure $\overline{\pi_0(D)}$ is compact in R_0 , we can find a simple loop on R_0 which is freely homotopic to b_0 and meets $R_0 \setminus \overline{\pi_0(D)}$. Lifting the loop and deforming the lift, we obtain a simple arc \tilde{b} on D_0 such that

- (i) the end points of \tilde{b} are projected to the same point by π_0 , and the image loop $\pi_0(\tilde{b})$ is freely homotopic to b_0 ,
- (ii) the arc \tilde{b} crosses \tilde{a}_0 once transversely,
- (iii) one of the end points of \tilde{b} is on ∂D and the other lies outside of D , and
- (iv) the part $\tilde{b}' := \tilde{b} \setminus D$ is connected.

We construct a marked once-holed torus $\tilde{X} = (\tilde{T}, \tilde{\chi})$ belonging to $\mathfrak{T}_a[Y_0]$ as follows. We start with $D \cup \tilde{b}$. By identifying the end points of \tilde{b} and thickening \tilde{b}' slightly and appropriately, we obtain a once-holed torus \tilde{T} so that π_0 induces a holomorphic mapping of \tilde{T} into R_0 . The curves \tilde{a}_0 and \tilde{b}_0 together make a mark $\tilde{\chi}$ of handle of \tilde{T} . It is obvious that $\tilde{X} = (\tilde{T}, \tilde{\chi})$ is an element of $\mathfrak{T}_a[Y_0]$. As D is a doubly connected domain on \tilde{T} with $\Gamma(D) \subset \Gamma(\tilde{X})$, we have

$$\lambda_a[Y_0] \leq \lambda(\tilde{X}) \leq \lambda(D) = \frac{1}{\pi} l(D) < \frac{1}{\pi} \{l(Y_0) + \varepsilon\}.$$

Since ε is arbitrary, we deduce that

$$\lambda_a[Y_0] \leq \frac{1}{\pi} l(Y_0),$$

which completes the proof of Theorem 3. \square

Next we evaluate the critical extremal length $\lambda_c[Y_0]$ for the existence of conformal mappings. The following theorem was announced in [10].

Theorem 4. $\lambda_c[Y_0] = \lambda(Y_0)$.

Proof. If there is a conformal mapping f of a marked once-holed torus X into Y_0 , then the image family $f_*(\Gamma(X))$ is included in $\Gamma(Y_0)$. Since conformal mappings keep extremal lengths invariant, it follows that $\lambda(X) \geq \lambda(Y_0)$ and hence that

$$\lambda_c[Y_0] \geq \lambda(Y_0).$$

To show that the sign of equality actually occurs, we employ results on Jenkins-Strebel differentials, that is, holomorphic quadratic differentials with closed trajectories (see Strebel [13, Chapter 5]). There uniquely exists a doubly connected domain Δ_0 on R_0 such that $\Gamma(\Delta_0) \subset \Gamma(Y_0)$ and $\lambda(\Delta_0) = \lambda(Y_0)$. It is dense in R_0 and is swept out by closed horizontal trajectories of a holomorphic quadratic differential on R_0 . Let $\delta_0 = \pi/\lambda(Y_0)$ and $r_0 = e^{\delta_0}$, and set $A(\delta) = \{z \in \mathbb{C} \mid e^{-\delta}r_0 < |z| < e^{\delta}r_0\}$ for $\delta \in (0, \delta_0]$. Then there is a conformal mapping F_0 of the annulus $A_0 := A(\delta_0)$ onto Δ_0 , which is continuously extended to a union E_0 of open arcs on ∂A_0 . We assume that E_0 is maximal with this property. Thus F_0 is a continuous mapping of $A_0 \cup E_0$ onto R_0 , and R_0 is obtained from $A_0 \cup E_0$ by identifying points on E_0 in the obvious manner (see Jenkins-Suita [3, Corollary 1 to Theorem 2]). Let a'_0 be the loop on R_0 corresponding to the circle $|z| = r_0$; we orient it so that it is freely homotopic to a_0 . Take a piecewise analytic simple loop b'_0 on R_0 freely homotopic to b_0 such that the intersection of $F_0^{-1}(b'_0)$ with the closure of a narrow annulus $A' := A(\delta_1)$ is a radial segment, where δ_1 is a sufficiently small positive number. By thickening b'_0 we obtain a doubly connected domain B' with b'_0 separating the boundary components of B' . We choose B' so that the union $T' := F_0(A') \cup B'$ is a once-holed torus included in R_0 . Obviously, $\chi' := \{a'_0, b'_0\}$ is a mark of handle of T' and the inclusion mapping $T' \rightarrow R_0$ is a conformal mapping of the marked once-holed torus (T', χ') into Y_0 . For each $\delta \in (0, \delta_0)$ take a homeomorphism h_δ of the interval $[1, r_0^2] = [1, e^{2\delta_0}r_0]$ onto itself such that $h_\delta(1) = 1$, $h_\delta(e^{-\delta_1}r_0) = e^{-\delta}r_0$ and $h_\delta(e^{\delta_1}r_0) = e^{\delta}r_0$. It induces a homeomorphism H_δ of R_0 onto itself satisfying $H_\delta \circ F_0(re^{i\theta}) = F_0(h_\delta(r)e^{i\theta})$. Intuitively, H_δ fattens $F_0(A')$ if $\delta > \delta_1$. The marked once-holed torus $X'_\delta := (H_\delta(T'), \chi'_\delta)$, where $\chi'_\delta = \{a'_0, H_\delta(b'_0)\}$, is conformally embedded into Y_0 . Thus $X'_\delta \in \mathfrak{T}_c[Y_0]$. Since $H_\delta(T')$ includes $F_0(A(\delta))$, we have

$$\lambda_c[Y_0] \leq \lambda(X'_\delta) \leq \lambda(A(\delta)) = \frac{\pi}{\delta}.$$

Letting $\delta \rightarrow \delta_0$, we obtain

$$\lambda_c[Y_0] \leq \frac{\pi}{\delta_0} = \lambda(Y_0).$$

This completes the proof. \square

Theorems 3 and 4 give a simple alternative proof of one of our previous results. The next corollary implies that $\mathfrak{T}_a[Y_0] \setminus \mathfrak{T}_c[Y_0]$ has a nonempty interior since $\mathfrak{T}_a[Y_0]$ and $\mathfrak{T}_c[Y_0]$ are closed domains with Lipschitz boundary.

Corollary 3 ([10, Theorem 8]). $\lambda_a[Y_0] < \lambda_c[Y_0]$.

Proof. Let Δ_0 be the doubly connected domain as in the proof of Theorem 4. Then

$$\lambda_a[Y_0] = \frac{1}{\pi}l(Y_0) < \frac{1}{\pi}l(\Delta_0) = \lambda(\Delta_0) = \lambda(Y_0) = \lambda_c[Y_0],$$

where the inequality follows from the fact that Δ_0 is a proper subdomain of R_0 . \square

Let $\mathfrak{T}_\infty[Y_0]$ be the set of $X \in \mathfrak{T}$ such that there is a holomorphic mapping $f : X \rightarrow Y_0$ with $d(f) < \infty$. Again, $\Pi \circ \Sigma(\mathfrak{T}_\infty[Y_0])$ is a horizontal strip. In fact, there is a nonnegative number $\lambda_\infty[Y_0]$ such that

- (i) if $\text{Im } \tau \geq 1/\lambda_\infty[Y_0]$, then $X_\tau^{(s)} \notin \mathfrak{T}_\infty[Y_0]$ for any $s \in [0, 1)$, while
- (ii) if $\text{Im } \tau < 1/\lambda_\infty[Y_0]$, then $X_\tau^{(s)} \in \mathfrak{T}_\infty[Y_0]$ for some $s \in [0, 1)$

(see [10, Theorem 4]). Theorem 3 together with [9, Theorem 2] yields the following identity:

Corollary 4. $\lambda_\infty[Y_0] = \frac{1}{\pi}l(Y_0)$.

Proposition 2 shows that the horizontal strip $\Pi \circ \Sigma(\mathfrak{T}_a[Y_0])$ never meets the critical horizontal line $\text{Im } \tau = 1/\lambda_a[Y_0]$. In other words, if $\text{Im } \tau = 1/\lambda_a[Y_0]$, then $X_\tau^{(s)}$ does not belong to $\mathfrak{T}_a[Y_0]$ for any $s \in [0, 1)$ (see [10, Theorem 6]). This is not always the case for the critical extremal lengths for the existence of conformal mappings. In fact, if Y_0 is a marked once-holed torus, then the strip $\Pi \circ \Sigma(\mathfrak{T}_c[Y_0])$ and the line $\text{Im } \tau = 1/\lambda_c[Y_0]$ intersect precisely at one point: there uniquely exists $\tau \in \mathbb{H}$ with $\text{Im } \tau = 1/\lambda_c[Y_0]$ such that $X_\tau^{(s)}$ belongs to $\mathfrak{T}_c[Y_0]$ for some $s \in [0, 1)$.

We show that there is also a Riemann surface Y_0 such that $\Pi \circ \Sigma(\mathfrak{T}_c[Y_0])$ does not meet the horizontal line $\text{Im } \tau = 1/\lambda_c[Y_0]$. To construct an example we give a preparatory consideration. Let $Y_0 = (R_0, \chi_0)$, where $\chi_0 = (a_0, b_0)$, be a Riemann surface with marked handle which is not a marked torus, and let Δ_0 be the (unique) doubly connected domain on R_0 with $\Gamma(\Delta_0) \subset \Gamma(Y_0)$ and $\lambda(\Delta_0) = \lambda(Y_0)$. Suppose that there is a conformal mapping f of a marked once-holed torus $X_\tau^{(s)}$ with $\text{Im } \tau = 1/\lambda_c[Y_0]$ into Y_0 . Since $\lambda(f_*(D_\tau)) = \lambda_c[Y_0] = \lambda(Y_0)$, we obtain $f(D_\tau) = \Delta_0$ by uniqueness. The horizontal arc $T_\tau^{(s)} \setminus D_\tau$ is mapped onto an arc γ_0 on the boundary $\partial\Delta_0$, and $f(T_\tau^{(s)})$ is identical with $\Delta_0 \cup \gamma_0$. This imposes a condition on b_0 , for, it is freely homotopic to $f_*(b_\tau^{(s)})$ on R_0 .

Example 1. Set $R_0 = T_{\tau_0} \setminus \pi_{\tau_0}(\{0, 1/2\})$, which is a twice-punctured torus. Let a_0 and b_0 be the projections of the segments $[\tau_0/2, 1 + \tau_0/2]$ and $[3/4, 3/4 + \tau_0]$, respectively. They are simple loops on R_0 , and make a mark χ_0 of handle of R_0 . Let $Y_0 = (R_0, \chi_0)$. Then $\lambda_c[Y_0] = \lambda(Y_0) = 1/\text{Im } \tau_0$. Since $X_{\tau_0}^{(1/2)}$ belongs to $\mathfrak{T}_c[Y_0]$, the strip $\Pi \circ \Sigma(\mathfrak{T}_c[Y_0])$ and the line $\text{Im } \tau = 1/\lambda_c[Y_0]$ meet at τ_0 .

Example 2. Let R_0 and a_0 be as in the preceding example. Let b'_0 be the projection of the polygonal arc obtained by joining the segments $[-1/4, \tau_0/4]$, $[\tau_0/4, 1/2 - \tau_0/4]$ and $[1/2 - \tau_0/4, 3/4 + \tau_0]$. Set $\chi'_0 = \{a_0, b'_0\}$ and $Y'_0 = (R_0, \chi'_0)$. Again, we have $\lambda_c[Y'_0] = 1/\text{Im } \tau_0$. However, $\Pi \circ \Sigma(\mathfrak{T}_c[Y'_0])$ does not meet the critical horizontal line $\text{Im } \tau = 1/\lambda_c[Y'_0]$.

REFERENCES

- [1] M. F. Bourque, The converse of the Schwarz lemma is false, *Ann. Acad. Sci. Fenn.* **41** (2016), 235–241.
- [2] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Birkhäuser, Boston-Basel-Berlin, 1992.
- [3] J. A. Jenkins and N. Suita, On analytic self-mappings of Riemann surfaces II, *Math. Ann.* **209** (1974), 109–115.
- [4] J. Kahn, K. M. Pilgrim and D. P. Thurston, Conformal surface embeddings and extremal length, preprint, arXiv:1507.05294.
- [5] A. Marden, I. Richards and B. Rodin, Analytic self-mappings of Riemann surfaces, *J. Anal. Math.* **18** (1967), 197–225.
- [6] B. Maskit, Comparison of hyperbolic and extremal lengths, *Ann. Acad. Sci. Fenn.* **10** (1985), 381–386.
- [7] M. Masumoto, Conformal mappings of a once-holed torus, *J. Anal. Math.* **66** (1995), 117–136.
- [8] M. Masumoto, Once-holed tori embedded in Riemann surfaces, *Math. Z.* **257** (2007), 453–464.

- [9] M. Masumoto, On critical extremal length for the existence of holomorphic mappings of once-holed tori, J. Inequal. Appl. 2013, **2013**:282.
- [10] M. Masumoto, Holomorphic mappings of once-holed tori, to appear in J. Anal. Math.
- [11] M. Shiba, The moduli of compact continuations of an open Riemann surface of genus one, Trans. Amer. Math. Soc. **301** (1987), 299–311.
- [12] M. Shiba, The euclidean, hyperbolic, and spherical spans of an open Riemann surface of low genus and the related area theorems, Kodai Math. J. **16** (1993), 118–137.
- [13] K. Strebel, Quadratic Differentials, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.
- [14] S. Wolpert, The length spectra as moduli for compact Riemann surfaces, Ann. of Math. **109** (1979), 323–351.

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